§ Unitary & orthogonal Operators

Recall: If $TT^* = T^*T = I$, then

- $F = \mathbb{C}$. T is called unitary operator
- $F = \mathbb{R}$. T is called orthogonal operator.

Theorem: V is finite-dim inner product space. $T \in \mathcal{L}(V)$.
Then the following statements are equivalent:

(a). $TT^* = T^*T = I$.

(b). T preserves the inner product on V. i.e., $\langle T(\vec{x}), T(\vec{y}) \rangle = \langle \vec{x}, \vec{y} \rangle$

(c). If β is an orthonormal basis for V, then $T(\beta)$ is an orthonormal basis.

(d). $\exists$ an orthonormal basis for V s.t. $T(\beta)$ is an orthonormal basis

(e). $\| T(\vec{x}) \| = \| \vec{x} \| \quad \forall \vec{x} \in V$.

Proof.

(a) $\Rightarrow$ (b). $\langle T(\vec{x}), T(\vec{y}) \rangle = \langle \vec{x}, \overset{= I}{\underline{T^* T}}(\vec{y}) \rangle = \langle \vec{x}, \vec{y} \rangle$

(b) $\Rightarrow$ (c). Let $\beta = \{\vec{v}_1, \cdots, \vec{v}_n\}$ be an orthonormal basis for V.

Then $\langle T(\vec{v}_i), T(\vec{v}_j) \rangle = \langle \vec{v}_i, \vec{v}_j \rangle = \delta_{ij}$

$\Rightarrow T(\beta) = \{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is an orthonormal basis.

(c) $\Rightarrow$ (d): trivial.

(d) $\Rightarrow$ (e): Take $\vec{x} \in V$. Let $\beta = \{\vec{v}_1, \cdots, \vec{v}_n\}$ be an orthonormal basis

Such that $T(\beta) = \{T(\vec{v_1}), \dots, T(\vec{v_n})\}$ is also orthonormal.

Suppose $\vec{x} = \sum_{i=1}^{n} a_i \vec{v_i}$, then $\|\vec{x}\|^2 = \sum_{i=1}^{n} |a_i|^2$

$$\|$$

$T(\vec{x}) = \sum_{i=1}^{n} a_i T(\vec{v_i})$, hence $\|T(\vec{x})\|^2 = \sum_{i=1}^{n} |a_i|^2$

(e) ⇒ (a). Want to show $U := I - T^*T$ is zero operator.

$$\langle T\vec{x}, T\vec{x} \rangle$$

Let $\vec{x} \in V$. then
$$\langle \vec{x}, U\vec{x} \rangle = \langle \vec{x}, (I - T^*T)\vec{x} \rangle$$
$$= \langle \vec{x}, \vec{x} \rangle - \langle \vec{x}, T^*T\vec{x} \rangle$$
$$= \|\vec{x}\|^2 - \|T(\vec{x})\|^2$$
$$= 0$$

Note: U is self-adjoint: $U^* = (I - T^*T)^* = I - T^*T = U$.

The lemma below implies $U = T_0$ the zero operator.

$\Rightarrow T^*T = I$. Since T is invertible, we also have $TT^* = I$. $\square$

Lemma: Let U be a self-adjoint operator on a finite-dim inner product space.

If $\langle \vec{x}, U\vec{x} \rangle = 0 \ \forall \vec{x} \in V$. Then $U = T_0$.

for either $F = \mathbb{C}$ or $\mathbb{R}$.

Pf: By the main Thm, $\exists$ an orthonormal basis β of eigenvectors of U

Let $\vec{x} \in \beta$, then $U(\vec{x}) = \lambda \vec{x}$ for some $\lambda \in F$.

and $0 = \langle \vec{x}, U(\vec{x}) \rangle = \langle \vec{x}, \lambda \vec{x} \rangle = \bar{\lambda} \|\vec{x}\|^2 = \bar{\lambda}$

$\Rightarrow \lambda = 0$.

$\Rightarrow U(\vec{x}) = 0 \cdot \vec{x} = 0 \quad \forall \vec{x} \in \beta$.

$\Rightarrow U = T_0$, the zero operator.

$\square$

Analogous Matrix Theory:

$A \in M_{n\times n}(\mathbb{C})$ is unitary if $A^* A = AA^* = I$.

$A \in M_{n\times n}(\mathbb{R})$ is orthogonal if $A^T \cdot A = AA^T = I$.

Notation: $U(n)$ = Set of $n\times n$ unitary matrices. $\subseteq M_{n\times n}(\mathbb{C})$

$O(n)$ = Set of $n\times n$ orthogonal matrices. $\subset M_{n\times n}(\mathbb{R})$.

Corollary: Let $A = \begin{array}{ccc} A\cdot e_1 & A\cdot e_2 & A\cdot e_n \\ \| & \| & \| \end{array}$

$$A = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix} \in M_{n\times n}(F)$$ where $\vec{v}_1, \cdots, \vec{v}_n$ are column vectors in F^n.

then A is unitary (resp. orthogonal)

iff $\beta = \{\vec{v}_1, \cdots, \vec{v}_n\}$ is an orthonormal basis for $\mathbb{C}^n$ (resp. $\mathbb{R}^n$)

Ex: Rotation matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in O(2)$

Def: Two matrices $A, B \in M_{n\times n}(\mathbb{C})$ are unitarily equivalent

if $\exists\, P \in U(n)$ s.t. $B = P^* A \cdot P$.

Rmk: • Note that $P^* = P^{-1}$ for $P \in U(n)$. so. $B = P^{-1} A P$.

$\Rightarrow$ A and B are similar.

• $B = P^* A P = P^{-1} A P$ $\Rightarrow$ $A = P B P^{-1}$

$= P B P^*$

$= (P^*)^* B P^*$

Theorem: Let $A \in M_{n\times n}(\mathbb{C})$. Then

unitarily diagonalizable

A is normal iff A is unitarily equivalent to a diagonal matrix

i.e., $\exists P \in U(n)$ s.t. P^*AP is diagonal.

Pf: ($\Rightarrow$): Suppose A is normal. Then $\exists$ orthonormal basis $\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$ for $\mathbb{C}^n$

s.t. $[L_A]_\beta$ is diagonal

$$= P^{-1}AP, \text{ where } P = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix}$$

Since β is orthonormal, P is unitary.

In particular $P^{-1} = P^*$. $\Rightarrow$ P^*AP is diagonal.

$(\Leftarrow)$ Assume $\exists\, P \in U(n)$ s.t. $P^*AP =: D$ is diagonal.

then $A = PDP^*$.

then
$$AA^* = (PDP^*)(PDP^*)^* = PD\overbrace{\cancel{P^*P}}^{I}D^*P^* = PDD^*P^*$$
$$A^*A = (PDP^*)^* PDP^* = PD^*\cancel{P^*P}DP^* = PD^*DP^*$$

Note $DD^* = D^*D$ as D is diagonal $\Rightarrow AA^* = A^*A$ $\square$

Similarly, for $F=\mathbb{R}$. we have

Theorem: Let $A \in M_{n\times n}(\mathbb{R})$, then

orthogonalility diagonalizable

A is symmetric iff A is orthogonally equivalent to a diagonal matrix.

i.e., $\exists P \in O(n)$ s.t. P^TAP is diagonal.

(pf: Exercise)

Example: $A \in \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$

Then $\exists\, P \in O(3)$ s.t. $P^T A P$ is diagonal.

Q: Find P explicitly.

- First find all eigenvalues & eigenvectors of A.

  $f_A(t) = (8-t)\cdot(2-t)^2$. eigenvalues: $\lambda = 2$ or 8.

- For $\lambda = 8$. $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector.

For $\lambda = 2$. $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for the eigenspace E_2

- Apply G.S process to get an orthogonal basis

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

- normalization:

$$\beta = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

· Thus $P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \in O(3)$

and $P^T \cdot A\, P = \begin{pmatrix} 2 & & \\ & 2 & \\ & & 8 \end{pmatrix}$

$\square$

§ Spectral Theorem

T normal $(F = \mathbb{C})$, self-adjoint $(F = \mathbb{R})$ operators

$\Longleftrightarrow$

$\exists$ orthonormal basis of eigenvectors

normal matrices (resp. symm matrices) unitarily (resp. orthog.) diagonalizable

$P^* A P$ diagonal, $P \in U(n)$

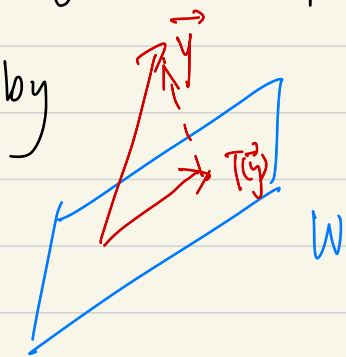
$\Longleftrightarrow$

Spectral Theorem: " $T = \text{stretch} \circ \text{Proj}$ "

Prop: Let V be an inner product space and $W \subset V$ is a finite-dim subspace with orthonormal basis $\{\vec{v}_1, \cdots, \vec{v}_k\}$.
Then the orthogonal projection $T: V \to V$ defined by

$$T(\vec{y}) = \sum_{i=1}^{k} \langle \vec{y}, \vec{v}_i \rangle \vec{v}_i$$

is linear operator. s.t.



(1) $N(T) = W^{\perp}$ and $R(T) = W$

(2). $T^2 = T$

(3). T is self-adjoint.

Pf (1) • $N(T) = \{ \vec{y} \in V ; \sum_{i=1}^{k} \langle \vec{y}, \vec{v_i} \rangle \vec{v_i} = 0 \}$

$= \{ \vec{y} \in V : \langle \vec{y}, \vec{v_i} \rangle = 0, \ \forall i = 1, \ldots, k \} = W^{\perp}$

Since $\{\vec{v_1}, \ldots, \vec{v_k}\}$ is a basis for W.

• To show $R(T) = W$. Clearly $R(T) \subset W_k$

On the other hand, $\forall \vec{u} \in W$. $\vec{u} = \sum_{i=1} \langle \vec{u}, \vec{v_i} \rangle \vec{v_i} = T(\vec{u})$.

So $R(T) = W$.

(2). Above argument shows $T|_W = I_W$. Hence, $T^2 = I_W T = T$.

(3). Take $\vec{x}, \vec{y} \in V$. Write $\vec{x} = \vec{x_1} + \vec{x_2}$, $\vec{y} = \vec{y_1} + \vec{y_2}$

s.t. $\vec{x_1}, \vec{y_1} \in W$, $\vec{x_2}, \vec{y_2} \in W^{\perp}$.

Then $\langle \vec{x}, T(\vec{y}) \rangle = \langle \vec{x_1} + \vec{x_2}, \vec{y_1} \rangle = \langle \vec{x_1}, \vec{y_1} \rangle$

$\langle T(\vec{x}), \vec{y} \rangle = \langle \vec{x_1}, \vec{y_1} + \vec{y_2} \rangle = \langle \vec{x_1}, \vec{y_1} \rangle$

So $\langle \vec{x}, T(\vec{y}) \rangle = \langle T(\vec{x}), \vec{y} \rangle$, then T is self-adjoint.

$\square$

Given $\vec{u}$, What is $T(\vec{u})$ assuming T has orthonormal basis of eigenvectors?

Ex: V orthonormal basis $\{\vec{v}_1, \vec{v}_2\}$. $T\vec{v}_i = \lambda_i \vec{v}_i$.

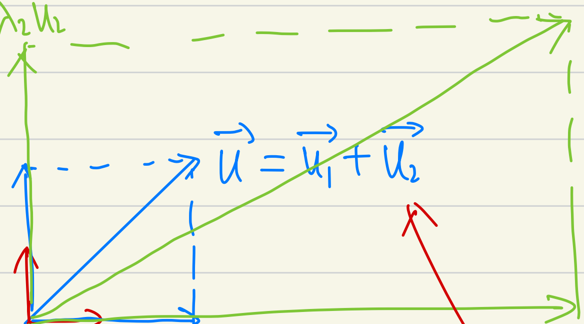
$T\vec{u}_2 = \lambda_2 \vec{u}_2$

$$T\vec{u} = \lambda_1 \vec{u}_1 + \lambda_2 \vec{u}_2$$

$$= (\lambda_1 \cdot Proj_1 + \lambda_2 \cdot Proj_2)\vec{u}$$

$T_2\vec{u} = Proj_2 \vec{u} = \vec{u}_2$

$\vec{u} = \vec{u}_1 + \vec{u}_2$



$$\vec{u}_1 = Proj_1 \vec{u}$$

$$= T_1 \vec{u}$$

$T\vec{u}_1 = \lambda_1 \vec{u}_1$

$$T = \lambda_1 T_1 + \lambda_2 T_2$$

$$I = T_1 + T_2.$$

Theorem: Let T be a linear operator on a finite-dim inner product space V over F with distinct eigenvalues $\lambda_1, \dots, \lambda_k$.

Assume that T is normal (resp. self-adjoint) if $F=\mathbb{C}$ (resp. $F=\mathbb{R}$)

For $i=1,\dots,k$, let $E_i = E_{\lambda_i}$ be the eigenspace of T corr. to λ_i and let T_i be the orthogonal projection onto E_i. Then:

(a). $V = E_1 \oplus \cdots \oplus E_k$.

(b). $E_i^{\perp} = \bigoplus_{j \neq i} E_j$ for $i=1,\dots,k$.

(c). $T_i T_j = \delta_{ij} T_j$ for $1 \le i,j \le k$

(d). $I = T_1 + \cdots + T_k$ ← Resolution of the identity operator.

(e) $T = \underbrace{\lambda_1}\, T_1 + \underbrace{\lambda_2}\, T_2 + \cdots + \underbrace{\lambda_k}\, T_k$ ← Spectral decomposition.

"spectrum" of T

Proof follows from:

- $\exists$ orthonormal basis of eigenvectors.
- Representation of $T(\vec{u})$ in the previous page.

$\square$